Defining Connectives and Expressive Adequacy

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1 Defining Connectives

Suppose you are asked to define a given connective in terms of other connectives. How do you proceed? Here we'll work an example first, and then we'll give a summary of the strategy.

Let's define $\rightarrow$ in terms of $\sim$ and $\wedge$. The first thing to do is to pick the simplest sentence that has only $\rightarrow$ in it: $(P \rightarrow Q)$. This is our *target sentence*. Our goal is going to be to find a sentence that's equivalent to our target sentence, but that uses only $\sim$ and $\wedge$. Once we have our target sentence, it generally helps to write the truth table for that sentence:

P	Q	$(P \rightarrow Q)$
T	T	T
T	F	F
F	T	T
F	F	T

At this point, the question you need to ask yourself is, "How do I say the same thing as 'If P, then Q' using only 'not' and 'and'?" At first, it's not at all obvious. However, one trick that can sometimes help is if you first try *negating* the target sentence, which in this case would give us $\sim (P \rightarrow Q)$. If we can express this negated target sentence using just conditional and negation, then all we have to do is add a negation to the front to express our original target sentence.

Okay, so how do we express $\sim (P \rightarrow Q)$ using just $\sim$ and $\wedge$? Well, here it turns out that we have a rule, NC, which applies to negated conditionals like $\sim (P \rightarrow Q)$. NC tells us that we can move from $\sim (P \rightarrow Q)$ to $(P \wedge \sim Q)$, *and vice versa*. Since NC lets us move in either direction between these two sentences, we know that these two sentences are logically equivalent. Sentences that are logically equivalent have the same truth table.[1] Let's write our new sentences in our truth table:

P	Q	$(P \rightarrow Q)$	$\sim (P \rightarrow Q)$	$(P \wedge \sim Q)$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

[1] If NC only let us move in one direction, say from $\sim (P \rightarrow Q)$ to $(P \wedge \sim Q)$ but not in reverse, then we couldn't conclude that those sentences are logically equivalent, and we'd need to look for a different sentence that *is* logically equivalent to $\sim (P \rightarrow Q)$.

From this truth table we can see that our sentence $(P \wedge \sim Q)$ has the opposite truth values of our target sentence, $(P \to Q)$. And we know how to flip a sentence's truth values: just add a negation to the front. Thus we get $\sim (P \wedge \sim Q)$. We can put this into our truth table to show that it is true in exactly the same situations as our target sentence $(P \to Q)$:

P	Q	$(P \to Q)$	$\sim (P \to Q)$	$(P \wedge \sim Q)$	$\sim (P \wedge \sim Q)$
T	T	T	F	F	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	F	F	T

And we're done: we've shown how to express $\to$ with just $\sim$ and $\wedge$. Any sentence of the form $(\square \to \bigcirc)$ can be rewritten as $\sim (\square \wedge \sim \bigcirc)$.

From this example, we can try to extract some general tips for expressing a given connective in terms of other connectives.

1) Select the simplest sentence that contains only the connective you want to express. This is your target sentence.

2) Make a truth table for your target sentence.

3) Your goal is now to find a sentence that's equivalent to your target sentence, but that uses only the connectives that you are trying to define it with. Here you should ask yourself, "How can I say the same thing using only the connectives that I'm allowed to use?" Often it helps here to try *negating* the target sentence first, and then asking this question, because we can always add a negation later.[2]

In step 3, if we have rules of inference for the connectives that you're using to define the target connective, then oftentimes these are a good place to look to find an equivalent sentence. If we don't have such rules, you'll either have to try to be creative and find a way of expressing the target connective using the connectives that you're allowed, or you can try playing around with truth tables until you get the result that you want, namely, a sentence that's logically equivalent to the target sentence that uses only the allowed connectives.

2 Expressive Adequacy

One context in which defining connectives is important is when you are trying to show that a set of connectives is expressively adequate. Recall that a set of connectives is *expressively adequate* if it can express every possible truth function.[3] In class, we proved that the set $\{\sim, \wedge, \vee\}$ is expressively adequate by showing that every truth function could be expressed by a sentence in Disjunctive Normal Form (DNF), and then noting that a sentence in DNF uses only negation, conjunction, and disjunction.

[2] That's not strictly speaking true. If you were asked to express a connective using a set of connectives that doesn't itself contain negation, then you might also have to figure out a way of expressing negation with that set.

[3] Some people use the name "truth-functional completeness" instead of "expressive adequacy." They mean the same thing.

Once we know that a set of connectives is expressively adequate, we can use this knowledge to prove that other sets of connectives are also expressively adequate. The strategy is to define the connectives from the set that we already know to be expressively adequate using the connectives from the set that we want to prove is expressively adequate.

For example, suppose the task was to show that $\{\uparrow\}$ is expressively adequate, given that $\{\sim, \rightarrow\}$ is expressively adequate. What we want to do, then, is to define both $\sim$ and $\rightarrow$ in terms of $\uparrow$. The thought is that, since $\{\sim, \rightarrow\}$ is expressively adequate, then if we can use $\uparrow$ to express anything that $\sim$ can express, and likewise if we can use $\uparrow$ to express anything that $\rightarrow$ can express, then it must be the case that $\{\uparrow\}$ can express anything that $\{\sim, \rightarrow\}$ can express. And since the latter set is expressively adequate, the former must be also.

What would happen if we went in reverse, and defined $\uparrow$ in terms of $\sim$ and $\rightarrow$? Well, then we'd be showing that the set $\{\sim, \rightarrow\}$ can express the truth function that is $\uparrow$. But we were already given that the set $\{\sim, \rightarrow\}$ is expressively adequate, meaning it can express *every* truth function, so of course it must be able to express $\uparrow$. So this wouldn't accomplish our goal of showing that $\{\uparrow\}$ is expressively adequate.