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# The Mechanical Method of Creating a Model Universe

## Overview

In predicate logic, it is trickier to show that an argument is invalid than it was in propositional logic. In propositional logic, we could just create a truth table, and show that it was possible for all of the premises to be true and the conclusion false. In predicate logic, we can't just make a truth table, because not all of the elements of our sentences are truth-functional. In particular, the *quantifiers* are not just a truth-function of the formulas within their scope. The truth-value of a quantified sentence depends on everything in the universe. So we need to create a model universe in order to show that it is possible for all of the premises to be true and the conclusion false. One way to do this is just by thinking about what the sentences mean in English, and building our universe to make all of the premises true and the conclusion false. Another way to do it is with the mechanical method.

## The Mechanical Method

Here is a general, step-by-step description of how to use the mechanical method to create a model universe. This tells us how to translate a sentence with quantifiers into a new sentence *without* quantifiers. This new sentence says the same thing about our universe as the original sentence did.

1. Determine the universe. The default universe is $\{0, 1\}$.
2. For any names occurring in your sentence, choose a particular object in the universe. This is the object that that name refers to. If there are no names, you can skip this step.
3. Scan your sentence for quantifiers. Figure out what the main quantifier is, and what variable it is binding.
4. Eliminate that quantifier.
5. Replace all occurrences of the variable that the quantifier was binding with the first member of the universe, then the second, then the third... until you've used all members of the universe. Don't change anything else about the rest of the sentence as you do this. In particular, be sure not to change the scope of the quantifier. *Only rewrite the part of the sentence that's within the scope of the quantifier that you eliminated in step 4.*

6. Once you've done step 5, you'll be left with a bunch of individual formulas. If the quantifier that you eliminated in step 4 was a universal quantifier ($\forall x, \forall y, \forall z$, etc.) you should put *conjunctions* between each of the individual formulas. If the quantifier was an existential quantifier ($\exists x, \exists y, \exists z$, etc.), you should put *disjunctions* between each of the individual formulas.

7. Repeat steps 3-6 for each of these individual formulas until they contain no more quantifiers.

8. Replace any names occurring in the resulting sentence with the object that it refers to, as you determined in step 2.

## Example

Here is an example of using the mechanical method to show that the following argument is invalid:

$$\forall x Fx \rightarrow \exists x Gx$$

$$\therefore Fa \rightarrow Ga$$

1. Universe: $\{0, 1\}$

2. There is a name occurring in our argument: $a$. Let's let $a$ refer to 0. We'll write this like so:

   $$a : 0$$

   (This choice is arbitrary. We could just as well have made $a$ refer to 1 here. But $a$ has to refer to exactly one object; $a$ is *not* like a predicate, which can have multiple objects in its extension.)

3. We'll start with our premise, and then we'll do the conclusion afterward. There are two quantifiers in our premise. There's no *main* quantifier (they're both subsidiary to the main connective, which is a conditional). Let's start with the universal quantifier, $\forall x$. This is binding the $x$ in "$Fx$", but it is not binding the $x$ in "$Gx$" — that's bound by the existential quantifier.

4. Here is our sentence with the universal quantifier eliminated:

   $$Fx \rightarrow \exists x Gx$$

5. Our universe contains two objects: 0 and 1. So, replacing that $x$ with first 0 and then 1, we get:

   $$(F0 \quad F1 \rightarrow \exists x Gx)$$

   Notice here that $F0$ and $F1$ are not actually formulas, because "0" and "1" are not officially symbols in our language. Also notice that we did *not* rewrite the sentence like this:

   $$(F0 \rightarrow \exists x Gx)\ (F1 \rightarrow \exists x Gx) \quad \Leftarrow \textbf{WRONG!}$$

We only rewrite the portion that is within the scope of the quantifier we removed.

6. We eliminated a universal quantifier in step 4, so we have to *conjoin* the results of step 5, getting us this:

$$(F0 \wedge F1 \to \exists x Gx)$$

7. We now have to repeat the process for the second quantifier in our sentence from step 6:

   (a) Eliminate the quantifier: $(F0 \wedge F1 \to Gx)$

   (b) Replace the variable it was binding with 0 and 1: $(F0 \wedge F1 \to G0 \;\; G1)$

   (c) *Disjoin* the results (because it was an existential quantifier): $(F0 \wedge F1 \to G0 \vee G1)$

   So we have ended with the sentence $(F0 \wedge F1 \to G0 \vee G1)$. Since our universe contains only 0 and 1, this sentence says the same thing about our universe as the original premise did: if everything is $F$, then something is $G$.

8. Now we need to repeat this entire process for the conclusion sentence. Fortunately, this will be faster, because the conclusion has no quantifiers in it. In fact, the only thing we need to do for the conclusion is replace the name $a$ with its referent, i.e. the number 0. (Remember, back in step 2 we stipulated that $a$ refers to 0.) So the conclusion becomes:

$$(F0 \to G0)$$

9. So we've translated our original argument into the following one:

$$(F0 \wedge F1 \to G0 \vee G1) \qquad \therefore (F0 \to G0)$$

   We now need to make the premise true and the conclusion false. Often it's easiest to start with the conclusion:

$$(F0 \wedge F1 \to G0 \vee G1) \qquad \therefore (F0 \underset{\mathbf{F}}{\to} G0)$$

   The conclusion is a conditional. To make a conditional false, we need to make the antecedent true and the consequent false:

$$(F0 \wedge F1 \to G0 \vee G1) \qquad \therefore (\underset{\mathrm{T}}{F0} \underset{\mathbf{F}}{\to} \underset{\mathrm{F}}{G0})$$

   Now we've made the conclusion false by setting F0 to be true and G0 to be false. We thus need to make sure that F0 gets assigned the value "T" wherever it occurs in our argument. Likewise we need to make sure that all occurrences of G0 gets assigned the value “F”:

$$(\underset{\mathrm{T}}{F0} \wedge F1 \rightarrow \underset{\mathrm{F}}{G0} \vee G1) \qquad \therefore (\underset{\mathrm{T}}{F0} \underset{\mathbf{F}}{\rightarrow} \underset{\mathrm{F}}{G0})$$

We need to ensure that the premise comes out true. The premise is also a conditional, so as long as we avoid having its antecedent be true and its consequent false, it will be true. So we have some choice about how we assign the remaining truth values. Let’s make F1 true and G1 true:

$$(\underset{\mathrm{T}}{F0} \wedge \underset{\mathrm{T}}{F1} \rightarrow \underset{\mathrm{F}}{G0} \vee \underset{\mathrm{T}}{G1}) \qquad \therefore (\underset{\mathrm{T}}{F0} \underset{\mathbf{F}}{\rightarrow} \underset{\mathrm{F}}{G0})$$

This allows us to determine the truth values of the antecedent and consequent of the premise, and thus we can determine the truth value of the entire premise:

$$(\underset{\mathrm{T}}{F0} \underset{\mathrm{T}}{\wedge} \underset{\mathrm{T}}{F1} \underset{\mathbf{T}}{\rightarrow} \underset{\mathrm{F}}{G0} \underset{\mathrm{T}}{\vee} \underset{\mathrm{T}}{G1}) \qquad \therefore (\underset{\mathrm{T}}{F0} \underset{\mathbf{F}}{\rightarrow} \underset{\mathrm{F}}{G0})$$

Here we have made the premise true and the conclusion false. We can read off the rest of our model universe from this step. Since $F0$ is true, we know that 0 is in the extension of $F$. $F1$ is also true, so 1 is in the extension of $F$. $G0$ is false, so 0 is *not* in the extension of $G$. $G1$ is true, so 1 is in the extension of $G$. So our model universe looks like this:

Universe: {0, 1}
$F$: {0, 1}
$G$: {1}
$a$: 0

Note that when your argument includes names, you need to include a specification of which objects those names refer to as part of your model universe.

As you can see, the mechanical method is fairly time-consuming, so if you can come up with a model universe just by thinking about the argument, that is often quicker. However, for more complicated arguments, it’s good to have this mechanical method to fall back on.